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Arrows of time and chaotic properties of the cosmic background radiation

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Abstract
We advance a new viewpoint on the connection between the thermodynamical and cosmological arrows of time, which can be traced via the properties of cosmic microwave background (CMB) radiation. We show that in the Friedmann–Robertson–Walker universe with negative curvature there is a necessary ingredient for the existence of the thermodynamical arrow of time. It is based on the dynamical instability of motion along null geodesics in a hyperbolic space. Together with special (de-correlated) initial conditions, this mechanism is sufficient for the thermodynamical arrow, whereas the special initial conditions alone are able to generate only a pre-arrow of time. Since the negatively curved space will expand forever, this provides a direct connection between the thermodynamical and cosmological arrows of time. The structural stability of the geodesic flows on hyperbolic spaces and hence the robustness of the proposed mechanism is especially stressed. We then point out that the main relations of equilibrium statistical thermodynamics (including the second law) do not necessarily depend on any arrow of time. Finally we formulate a *curvature anthropic principle*, which stipulates the negative curvature as a necessary condition for the time-asymmetric universe with an observer. CMB has to carry the signature of this principle as well.

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1. Introduction

The structure and mechanisms of time asymmetry, or the arrows of time, remain among the much debated questions of modern physics, in spite of the intense attention devoted to these topics over many years. The relation between the thermodynamical and the cosmological arrows has been studied from very different viewpoints, e.g. by Gold, Penrose, Hawking [1–4], Page [5], Petrosky and Prigogine [6], Zeh and others [7, 8].

In this paper we argue that the cosmological arrow of time, as defined by the expansion of the universe, and a crucial observational fact on the existence of highly isotropic cosmic microwave background (CMB) radiation with the Planckian spectrum, can be connected with the thermodynamical arrow of time; preliminary accounts of these views are in [9]. We will argue that the curvature of the universe can have an essential role in this problem. CMB is a cornerstone of our discussion [10]. CMB photons are moving freely during almost the entire lifetime of the universe, thus tracing its geometry. Indeed, the properties of the CMB in a hyperbolic Friedmann–Robertson–Walker (FRW) ($k = -1$) universe differ from those of flat, $k = 0$, and positively curved, $k = +1$, cases.

In particular, the exponential deviation of the geodesics and the ensuing effect of geodesic mixing in the hyperbolic spaces lead at least to the following observable consequences [12, 13]:

(1) damping of anisotropy after the last scattering epoch,
(2) flattening of the autocorrelation function and
(3) distortion of anisotropy spots.

The statistically significant signature of the third effect—the threshold-independent ellipticity in the CMB sky maps—has been detected for COBE-DMR 4-year data [14]. Interpreted as a result of the geodesic mixing of photons, as predicted [13], this will model-independently indicate the negative curvature of the universe and $\Omega_0 < 1$. A more advanced descriptor to trace the curvature is the Kolmogorov complexity of CMB anisotropies [15, 16].

The recent Boomerang [17], Maxima, DASI, CPI (see [18]) data on the CMB power spectrum are interpreted as supporting the CDM flat models with adiabatic scale-invariant fluctuation spectrum, though there are claims for other models as well, see e.g. [19–21]. The precise flatness, however, cannot be proved due not only to the measurement errors but mainly to the degeneracy and dependence on a number of free parameters.

It is remarkable that threshold-independent behaviour of the ellipticity of the anisotropies has been found also for Boomerang maps [22] which differ from COBE-DMR maps not only in their higher angular resolution, but also due to their lower noise level. The supernovae data, supporting the existence of a cosmological term, are not decisive for the sign of the curvature [23, 24].

Let us now turn to the time asymmetry of the universe. Several different and, at first glance, independent time-arrows have been defined [2].

(i) Thermodynamical: entropy of a closed system increases with time.
(ii) Cosmological: the universe expands.
(iii) Psychological: knowledge of the past but not of the future.
(iv) Electromagnetic: retarded interaction (propagation).
(v) Quantum-mechanical: the change of a wavefunction during a typical measurement process is irreversible, i.e. neither unitary nor linear.

The electromagnetic and psychological arrows are viewed as a consequence of the thermodynamical arrow [2, 3, 8, 25]. Following, in particular, Landau and Lifshitz [11], some researchers may think that the quantum-mechanical arrow is independent of the others, and may even serve as a base for them. In contrast, we believe that there is no fundamental quantum-mechanical arrow of time, and the problem of quantum measurement can be fully explained within quantum statistical mechanics, i.e. essentially as a consequence of the thermodynamical arrow of time [26]. Note also that we did not involve here the CP-asymmetry of weak interactions, since it is not directly relevant to the present discussion [2].

Finally, only the thermodynamical and cosmological arrows are basic for our present purposes. Broadly speaking, the thermodynamical arrow for a statistical system can be formulated as a consequence of the following *necessary* conditions:

(1) de-correlated (special) initial conditions and
(2) no-memory dynamics.

Depending on their background people are sometimes inclined to overestimate one of those reasons, thereby underestimating the other. However, it should be emphasized once more that *both of them* are strictly necessary, as we show below. The above two conditions already appear in Boltzmann's derivation of the kinetic equation, though perhaps not explicitly. They can be traced out clearly in Zwanzig's derivation of the master-equation [27] or Jaynes' information-theoretical approach to irreversibility [28]. A conventional discussion about possible relations between cosmological and thermodynamical arrows of time concentrates on only the first condition [2–5, 8, 25].

One of our main intentions in this paper is to show that this is not sufficient, because special initial conditions alone can generate only a thermodynamical pre-arrow of time. Our main purpose here is to point out that along with the initial conditions, the second ingredient of the thermodynamical arrow can have a cosmological context as well, which arises due to the mixing of trajectories in hyperbolic spaces.

Namely, if the Friedmann–Robertson–Walker universe has a negative curvature, then the flow of null geodesics which describes the free motion of photons represents an Anosov system [29], a class of dynamical systems with maximally strong statistical properties. Anosov systems are characterized by an exponential divergence of initially close trajectories, by the property of K-mixing, positive Kolmogorov–Sinai (KS) entropy and countable Lebesgue spectrum. In particular, geodesic flows on compact manifolds with negative constant curvature are characterized by the exponential decay of the correlators [30]. One of the significant properties of Anosov systems is their structural stability, namely resistance of properties with respect to perturbations. This is crucial, since we do not live in a universe with strongly constant curvature but with small perturbations of the metric, and moreover, we know the magnitude of their smallness from the same CMB measurements.

On the other hand, sufficiently fast decoupling of correlations is responsible for the so-called Markovian behaviour [33, 36, 37]. As we shall discuss below, the latter property can be one of the two main ingredients ensuring the thermodynamical arrow, though the existence of other mechanisms is not excluded.

Thus we will show that the exponential instability of geodesic flow in the hyperbolic FRW universe—the geodesic mixing—revealed through the properties of CMB, relates the thermodynamical and cosmological arrows of time. The proposed mechanism does not offer an answer to an ambiguous question, what will happen to the thermodynamical arrow if the cosmological arrow were to be inverted, just because in the negatively curved space the cosmological arrow will never be inverted. One notes that many considerations and speculations on this question implicitly identify the thermodynamical arrow of time with the second law of thermodynamics and the appearance of a Gibbs distribution. In this context we will show that the second law and the Gibbs distribution can be obtained from purely time-symmetric arguments, and need not be consequences of the thermodynamical arrow. Further clarification is needed to understand the link of cosmology with concrete aspects of statistical physics.

The paper is organized as follows. In section 2 we discuss the thermodynamical arrow and the conditions which are necessary for its derivation. In section 3 we show how the CMB mixing properties in the negatively curved space can be connected with the thermodynamical arrow. Then we discuss the derivation of the second law and the Gibbs distribution in a way that does not depend on the thermodynamical arrow of time. Our conclusions are presented in the last section.

2. Thermodynamical arrow of time

2.1. Pre-arrow of time

In this section we will discuss the following aspects of the thermodynamical arrow of time. Starting from the standard system–bath approach we will indicate how the choice of only special initial conditions leads to a thermodynamical *pre-arrow* of time. This is not sufficient to generate the full thermodynamical arrow. Under suitable dynamical conditions related to the features of the bath, the pre-arrow generates the full thermodynamical arrow, namely a monotonic change with time of the proper thermodynamical potential, which is most typically entropy or free energy. In general, the arrow is present only for a relatively later stage of the relaxational dynamics of the system. Most frequently, this stage is connected with the Markovian properties. However, in certain special situations the thermodynamical arrow may also be present in a non-Markovian situation. Such a presentation of the thermodynamical arrow will, in particular, make clear that in general *both* dynamical features and special initial conditions are necessary for establishing the thermodynamical arrow of time.

For the sake of generality and simplicity we will work within the quantum-mechanical formalism. The reader can just keep in mind that all our results are directly transportable to classical physics upon changing density matrices to probability distributions, traces to integration over the phase space and the von Neumann equation to the Liouville equation.

Let a quantum system S interact with a thermal bath B. The total Hamiltonian is

$$H = H_{\mathrm{S}} + H_{\mathrm{B}} + H_{\mathrm{I}}. \tag{2.1}$$

We denote by H_{S} and H_{B} the Hamiltonians of the system and the bath respectively, whereas H_{I} stands for the interaction Hamiltonian. The state of the full system is described by the density matrix $\mathcal{D}(t)$, which satisfies the corresponding von Neumann equation

$$\mathrm{i}\partial_t \mathcal{D}(t) = [H, \mathcal{D}(t)] \qquad \mathcal{D}(t) = \mathrm{e}^{-\mathrm{i}tH/\hbar}\mathcal{D}(0)\,\mathrm{e}^{\mathrm{i}tH/\hbar} \tag{2.2}$$

where $[\,\cdot\,,\,\cdot\,]$ stands for the commutator as usual. The crucial assumption on the initial state can be formulated as follows:

$$\mathcal{D}(0) = D_{\mathrm{S}}(0) \otimes D_{\mathrm{B}}(0) \tag{2.3}$$

which means that at the initial time $t = 0$ the system and the bath were completely independent. The state of the system at arbitrary positive time t is described by the corresponding partial density matrix:

$$D_{\mathrm{S}}(t) = \mathrm{tr}_{\mathrm{B}}\mathcal{D}(t) \tag{2.4}$$

where tr_{B} indicates the trace over the Hilbert space of the bath. The important point of the system–bath approach—as well as any statistical physics approach which derives the thermodynamical arrow—is its dependence on incomplete observability: although the system and the bath constitute a closed system, one is interested in the state of the system only, which under the presence of the bath evolves according to a non-unitary dynamics generated by a superoperator $\mathcal{T}$:

$$D_{\mathrm{S}}(t) = \mathcal{T}(t,0)D(0) = \sum_{\alpha\beta} A_{\alpha\beta}D(0)A^{\dagger}_{\alpha\beta} \tag{2.5}$$

where $A_{\alpha\beta}$ are operators in the Hilbert space of S. They are determined via the spectral decomposition of the initial density matrix of the bath

$$D_{\mathrm{B}}(0) = \sum_{\alpha} \lambda_\alpha |\alpha\rangle\langle\alpha| \qquad \langle\alpha|\beta\rangle = \delta_{\alpha\beta} \tag{2.6}$$

where Kronecker $\delta_{\alpha\beta} = 1$ (0) for $\alpha = \beta$ ($\alpha \neq \beta$), and by the evolution operator generated by the complete Hamiltonian H

$$A_{\alpha\beta} = \sqrt{\lambda_\beta}\langle\alpha|\,\mathrm{e}^{-\mathrm{i}tH/\hbar}|\beta\rangle. \tag{2.7}$$

Equations (2.5) and (2.7) are easily obtained from (2.2) and (2.4) upon substituting (2.6) there. One can check directly that

$$\sum_{\alpha\beta} A^\dagger_{\alpha\beta} A_{\alpha\beta} = 1 \tag{2.8}$$

as required for the trace conservation of $D_{\mathrm{S}}(t)$ at any time. Three important facts should be noted in the context of (2.5): (i) the superoperator $\mathcal{T}$ appearing in (2.5) is not unitary, and in general it does not have an inverse operator. Thus, the dynamics of the system alone is irreversible. This is a consequence of the general fact that statistical systems are described incompletely, e.g., in the system–bath approach one focuses on the system alone in the presence of the bath. (ii) The operators $A_{\alpha\beta}$ do not depend on the initial state of the system itself. Thus, the dynamics of the system is *autonomous*, solely due to the initial condition (2.3). It is obvious that this property will not be valid for an arbitrary initial state. (iii) In general, the property $\mathcal{T}(t_f, t_i) = \mathcal{T}(t_f - t_i)$ for all $t_f > t_i$, which is automatically valid for the unitary situation, is broken inasmuch as the bath is present.

It appears that equations (2.2) and (2.3) are enough to ensure the existence of the pre-arrow, which is not a statement on the dynamics of the system itself but rather a statement on the similarity between the dynamical processes given by (2.5) and those generated by the same Hamiltonian (2.1) and somewhat different initial condition:

$$\mathcal{R}(0) = R_{\mathrm{S}}(0) \otimes D_{\mathrm{B}}(0). \tag{2.9}$$

Note that the difference between (2.3) and (2.9) is only in the initial condition for the system itself: $R_{\mathrm{S}}(0) \neq D_{\mathrm{S}}(0)$. An important measure of the difference between $R_{\mathrm{S}}(0)$ and $D_{\mathrm{S}}(0)$ is the relative entropy [39]:

$$S[D_{\mathrm{S}}(0)\|R_{\mathrm{S}}(0)] = \mathrm{tr}[D_{\mathrm{S}}(0)\ln D_{\mathrm{S}}(0) - D_{\mathrm{S}}(0)\ln R_{\mathrm{S}}(0)]$$

which is known to be non-negative and is equal to zero only for $R_{\mathrm{S}}(0) = D_{\mathrm{S}}(0)$. In general, the relative entropy $S[D_{\mathrm{S}}(0)\|R_{\mathrm{S}}(0)]$ characterizes the information needed to distinguish between the density matrices $D_{\mathrm{S}}(0)$ and $R_{\mathrm{S}}(0)$ via many ($\gg$1) independent identical experiments [41]. The fundamental theorem [37, 40, 41] states that at all later times the relative entropy between the density matrices $R_{\mathrm{S}}(t) = \mathrm{tr}_{\mathrm{B}}\mathcal{R}(t)$ and $D_{\mathrm{S}}(t) = \mathrm{tr}_{\mathrm{B}}\mathcal{D}(t)$ does not increase:

$$\begin{aligned} S[D_{\mathrm{S}}(0)\|R_{\mathrm{S}}(0)] &\geqslant S[D_{\mathrm{S}}(t)\|R_{\mathrm{S}}(t)] \\ &\equiv S[\mathcal{T}(t,0)D_{\mathrm{S}}(0)\|\mathcal{T}(t,0)R_{\mathrm{S}}(0)]. \end{aligned} \tag{2.10}$$

The equality sign in (2.10) is realized for unitary evolution showing that there is no pre-arrow for a closed system. Equation (2.10) shows that any dynamics for the system with the initial condition (2.3) does not increase the distinguishability between different initial conditions.

2.2. Arrow of time

Two additional dynamical conditions which lead to the appearance of the thermodynamical arrow of time are the following: (1) features of the bath are such that for sufficiently large t one has

$$\mathcal{T}(t,0) = \mathcal{T}(t) \tag{2.11}$$

i.e. the dependence on the initial time disappears (no-memory). (2) The system relaxes with time to a certain stationary density matrix $D_{\mathrm{S}}^{(st)}$:

$$\mathcal{T}(t) D_{\mathrm{S}}^{(st)} = D_{\mathrm{S}}^{(st)} \qquad D_{\mathrm{S}}(t) \to D_{\mathrm{S}}^{(st)}. \tag{2.12}$$

For times where (2.11) is valid, one can apply (2.10) for any θ as

$$\begin{aligned} S\left[D_{\mathrm{S}}(t) \| D_{\mathrm{S}}^{(st)}\right] &\geqslant S\left[\mathcal{T}(\theta) D_{\mathrm{S}}(t) \| \mathcal{T}(\theta) D_{\mathrm{S}}^{(st)}\right] \\ &\equiv S\left[D_{\mathrm{S}}(t+\theta) \| D_{\mathrm{S}}^{(st)}\right] \end{aligned} \tag{2.13}$$

and deduce that the function $S\big[D_{\mathrm{S}}(t) \| D_{\mathrm{S}}^{(st)}\big]$ is monotonically decreasing with time, since $\theta > 0$ was arbitrary. The concrete properties of this function depend on the structure of $D_{\mathrm{S}}^{(st)}$. If, for example, the stationary distribution is microcanonical: $D_{\mathrm{S}}^{(st)} \propto 1$, then (2.13) reduces to the statement that von Neumann entropy

$$S_{vN}[D_{\mathrm{S}}(t)] = -\mathrm{tr}[D_{\mathrm{S}}(t) \ln D_{\mathrm{S}}(t)]$$

increases with time. In the case of the canonical distribution function $D_{\mathrm{S}}^{(st)} \propto \exp(-H_{\mathrm{S}}/T)$, where T is temperature, one gets that the free energy

$$F = U(t) - T S_{vN}(t) \qquad U(t) = \mathrm{tr}[D_{\mathrm{S}}(t) H_{\mathrm{S}}]$$

monotonically decreases with time. Here $U(t)$ is the average energy; if it is conserved during evolution, then the statements on free energy and entropy are essentially equivalent. Note the difference with the pre-arrow of time which compares only the initial relative entropy $S[D_{\mathrm{S}}(0) \| R_{\mathrm{S}}(0)]$ with the relative entropy $S[D_{\mathrm{S}}(t) \| R_{\mathrm{S}}(t)]$ at any time $t > 0$, without making any connection between $S[D_{\mathrm{S}}(t) \| R_{\mathrm{S}}(t)]$ and $S[D_{\mathrm{S}}(t+\theta) \| R_{\mathrm{S}}(t+\theta)]$ for $\theta > 0$.

To summarize this subsection, we note that conditions (2.11) ensuring the appearance of the thermodynamical arrow of time is only sufficient; in certain situations it can be substituted by other (weaker) dynamical assumptions. The relaxation (2.12) should be, of course, understood on times much less than the Poincaré recurrent time. Therefore, the recurrent time itself must be very large. This condition can be considered as satisfied, since for majority of 'reasonable' systems the Poincaré time exceeds the age of the universe.

2.3. A scenario for the no-memory regime

In this section we will discuss a possible scenario for the appearance of condition (2.11). It is based on a sufficiently weak coupling between the system and the bath, as well as on the fast relaxation of the bath correlation functions. This last feature is intrinsic for the bath, and (for the considered limit) it has nothing to do with the coupling to the system. Starting from (2.1), it is convenient to introduce Liouville (super)operators:

$$\mathcal{L}_k(t) = \frac{1}{\mathrm{i}\hbar}[H_k, \ldots] \qquad k = \mathrm{S}, \mathrm{B} \tag{2.14}$$

$$\mathcal{L}_{\mathrm{I}}(t) = \frac{1}{\mathrm{i}\hbar}[H_{\mathrm{I}}(t), \ldots] \tag{2.15}$$

where $H_I(t)$ is the corresponding Heisenberg operator in the free representation (i.e. without the interaction):

$$H_I(t) = e^{\frac{it}{\hbar}(H_S+H_B)} H_I \, e^{-\frac{it}{\hbar}(H_S+H_B)}. \tag{2.16}$$

Using Liouville operators, the full dynamics for the overall density matrix $\mathcal{D}(t)$ is written as

$$\mathcal{D}(t) = e^{t(\mathcal{L}_S+\mathcal{L}_B)} \hat{T} \, e^{\int_0^t d\theta \mathcal{L}_I(\theta)} D_B(0) \otimes D_S(0) \tag{2.17}$$

where $\hat{T}$ is the time-ordering operator, and where $D_B(0)$ and $D_S(0)$ refer to the initial density matrices of the bath and the system, respectively. The marginal density matrix of the system, $D_S(t) = \mathrm{tr}_B \mathcal{D}(t)$, reads

$$D_S(t) = e^{t\mathcal{L}_I} \left\langle \hat{T} \, e^{\int_0^t d\theta \mathcal{L}_I(\theta)} \right\rangle D(0) \tag{2.18}$$

where for any quantity $\mathcal{X}$ (possibly a superoperator)

$$\langle \mathcal{X} \rangle \equiv \mathrm{tr}_B[\mathcal{X} D_B(0)]. \tag{2.19}$$

Note that the mutual ordering between $\mathcal{X}$ and $D_B(0)$ can be important. By analogy with the classical cumulant expansion one writes [34]

$$\left\langle \hat{T} \, e^{\int_0^t d\theta \mathcal{L}_I(\theta)} \right\rangle = \hat{T} \, e^{\int_0^t d\theta \mathcal{F}(\theta)} \tag{2.20}$$

where $\mathcal{F}$ is another superoperator, which is determined step-by-step by expanding both sides of (2.20) over H_I:

$$\mathcal{F} = \sum_{k=1} \mathcal{F}_k$$

with

$$\left\langle \hat{T} \, e^{-\frac{i}{\hbar} \int_0^t d\theta \mathcal{L}_I(\theta)} \right\rangle = 1 + \sum_{k=1} \int_0^t d\theta_1 \int_0^{\theta_1} d\theta_2 \cdots \int_0^{\theta_{k-1}} d\theta_k \langle \mathcal{L}_I(\theta_1) \cdots \mathcal{L}_I(\theta_k) \rangle. \tag{2.21}$$

The first two contributions to $\mathcal{F}$ are the following:

$$\mathcal{F}_1(t) = \langle \mathcal{L}_I(t) \rangle \qquad \mathcal{F}_2(t) = \int_0^t d\theta \, [\langle \mathcal{L}_I(t)\mathcal{L}_I(\theta) \rangle - \langle \mathcal{L}_I(t) \rangle \langle \mathcal{L}_I(\theta) \rangle] \, . \tag{2.22}$$

The content of the considered approximation is that one keeps only these two terms for $\mathcal{F}$, thus neglecting all other cumulants. This is applicable if the magnitude of H_I is sufficiently small. The cumulant expansion also ensures that possible secular terms are absent, so that the neglected higher order cumulants are typically homogeneously small compared with the second one [34]. Let us assume for simplicity that

$$\mathcal{F}_1(t) = 0 \tag{2.23}$$

and then the final differential convolutionless equation for $D_S(t)$ reads from (2.18):

$$\dot{D}_S(t) = \frac{1}{i\hbar}[H_S(t), D_S(t)] + e^{t\mathcal{L}_S} \mathcal{F}_2(t) \, e^{-t\mathcal{L}_S} D_S(t). \tag{2.24}$$

Note that this is a differential, though non-Markovian, equation for $D(t)$. So it is consonant with a thermodynamical pre-arrow of time, but may be compatible with a non-monotonic change of the corresponding thermodynamical potential. To implement the no-memory approximation, we will work out a particular case, where the initial interaction Hamiltonian is presented as

$$H_I = S \otimes B \tag{2.25}$$

with S and B belonging to the Hilbert spaces of the system and the bath, respectively. Then equation (2.24) reads

$$\dot{D}_{\mathrm{S}}(t) = \frac{1}{\mathrm{i}\hbar}[H_{\mathrm{S}}, D_{\mathrm{S}}(t)] - \frac{1}{\hbar^2}\int_0^t \mathrm{d}\theta\{K(t,\theta)(SS(\theta - t)D_{\mathrm{S}}(t) - S(\theta - t)D_{\mathrm{S}}(t)S) + \mathrm{h.c.}\} \tag{2.26}$$

$$S(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{S}}/\hbar} S\, \mathrm{e}^{-\mathrm{i}tH_{\mathrm{S}}/\hbar} \tag{2.27}$$

$$B(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}/\hbar} B\, \mathrm{e}^{-\mathrm{i}tH_{\mathrm{B}}/\hbar} \tag{2.28}$$

where $S(t)$ and $B(t)$ are free Heisenberg operators of the bath (i.e. they evolve under the uncoupled system and bath dynamics), h.c. before the end of the curly bracket means the Hermitian conjugate of the whole expression contained in this bracket, and where

$$K(t,\theta) = \langle B(t)B(\theta)\rangle \equiv \mathrm{tr}[B(t)B(s)D_{\mathrm{B}}(0)] \tag{2.29}$$

is the *free* correlation function of the bath variables.

Now assume that the decoupling time τ of the correlation function $K(t,\theta)$ is the smallest characteristic time of the considered situation:

$$K(t,\theta) \simeq \langle B(t)\rangle\langle B(\theta)\rangle = 0 \qquad \text{for} \quad |t - s| \gg \tau \tag{2.30}$$

where the last equality is realized due to (2.23), which in the present context reads: $\langle B(t)\rangle = \langle B(\theta)\rangle = 0$. This means that for $t \gg \tau$ the relevant integration domain of the integrals over θ in (2.26) is $\theta \ll t$, and the upper limit of these integrals can be substituted by infinity. Then the whole operator acting on $D_{\mathrm{S}}(t)$ can be viewed as t-independent. Thus, the solution of (2.26) is represented as

$$D_{\mathrm{S}}(t) = \exp[t\mathcal{L}_{\mathrm{eff}}]\, D_{\mathrm{S}}(0), \tag{2.31}$$

with an effective Liouville operator $\mathcal{L}_{\mathrm{eff}}$ obtained from (2.26). This is just the desired form (2.11). Thus, provided that the stationary distribution $D_{\mathrm{S}}^{(st)}$ exists, properties (2.11) and (2.12) are satisfied, and the thermodynamical arrow of time has been established as follows from equation (2.13).

The decoupling property (2.30) is seen to be connected with the dynamics of the free bath, see (2.28), and hence needs a concrete physical mechanism for its validity. The standard mechanism for this is to take a very large bath, consisting of many nearly independent pieces. Another possible mechanism is the intrinsic chaoticity of the bath, which leads to decoupling of correlators [35, 36, 42]. This property will be discussed in the next section.

3. Geodesics mixing

The geodesics of a space (locally if the space is non-compact) with constant negative curvature k in all two-dimensional directions are known to possess properties of Anosov systems.

The Jacobi equation which describes the deviation $\mathbf{n}$ of close geodesics

$$\frac{\mathrm{d}^2\mathbf{n}}{\mathrm{d}\lambda^2} + k\mathbf{n} = 0 \tag{3.1}$$

for $k = -1$ has the solution

$$\mathbf{n} = \mathbf{n}(\mathbf{0})\cosh\lambda + \dot{\mathbf{n}}(0)\sinh\lambda. \tag{3.2}$$

It was proved [30] (see also [31]) that for a dim $= 3$ compact manifold M with constant negative curvature the time correlation function of the geodesic flow $\{f^{\lambda}\}$ on the unit tangent bundle SM of M

$$b_{A_1,A_2}(\lambda) = \int_{SM} A_1(f^{\lambda}x)A_2(x)\,\mathrm{d}\mu - \int_{SM} A_1(x)\,\mathrm{d}\mu \int_{SM} A_2(x)\,\mathrm{d}\mu \tag{3.3}$$

decays exponentially for all functions $A_1, A_2 \in L^2(SM)$

$$\left|b_{A_1,A_2}(\lambda)\right| \leqslant c \cdot \left|b_{A_1,A_2}(0)\right| \cdot \mathrm{e}^{-h\lambda} \tag{3.4}$$

where $c > 0$, μ is the Liouville measure and $\mu(SM) = 1$, h is the KS (Kolmogorov–Sinai) entropy of the geodesic flow $\{f^\lambda\}$. To reveal the properties of the free motion of photons in pseudo-Riemannian $(3+1)$-space the projection of its geodesics into Riemannian 3-space has to be performed, i.e. by corresponding a geodesic $c(\lambda) = x(\lambda)$ to the geodesic in the former space: $\gamma(\lambda) = (x(\lambda), t(\lambda))$. Then the transformation of the affine parameter is as follows [32]:

$$\lambda(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{a(s)}.$$

The KS-entropy in the exponential index can be easily estimated for the matter-dominated post-scattering universe [12], so that

$$\mathrm{e}^{h\lambda} = (1+z)^2 \left[\frac{1+\sqrt{1-\Omega}}{\sqrt{1+z\Omega}+\sqrt{1-\Omega}}\right]^4 \tag{3.5}$$

i.e. depends on the density parameter Ω and the redshift of the last scattering epoch z. The initial condition (2.3) should ensure the thermodynamical arrow. The decay of correlators for geodesic flow for a $k = -1$ FRW universe provides the procedure of coarse-graining and ensures the Markovian (no-memory) behaviour of the CMB parameters, so that

$$t \gg \tau = 1/h \tag{3.6}$$

where the KS-entropy defines the characteristic timescale τ and depends only on the diameter of the universe which is the only scale in the maximally symmetric space [12]. The timescale τ is the so-called Markov time or a random variable independent of the future as defined in the theory of Markov processes [38], and in the CMB problem it describes the decay of initial perturbations, i.e. damping of the initial anisotropy amplitude and the flattening of the angular correlation function [12, 13]. For certain dynamical systems the timescale also defines the relaxation time for tending to a microcanonical equilibrium. The evaluation of τ and hence of the negative curvature of the universe has been performed in [14] using the COBE data. The negative constant curvature leads to a decay of time correlators of geodesics, thus defining the thermodynamic arrow for CMB in a FRW $k = -1$ universe.

4. Gibbs distribution, the second law and the arrow of time

The purpose of the present section is to discuss to what extent irreversibility and the thermodynamical arrow of time are *necessary* to establish the second law, and the Gibbs distribution, which are known to be the basis of equilibrium statistical mechanics. The development of statistical thermodynamics during the last century safely confirms the sufficiency of the thermodynamical arrow of time to derive the Gibbs distribution and the second law [11, 28]. To show that these are not necessary, we shall consider an alternative derivation of the Gibbs distribution proposed by Lenard [43]. Similar ideas were expressed in [44]. For a recent extension of these results see [45].

A closed statistical system is considered. Its dynamics is described by a Hamiltonian H. At the moment $t = 0$, where the state of the system is $D(0)$, an external time-dependent field is switched on, and the Hamiltonian becomes $\mathcal{H}(t)$. The field is switched off at the moment t, and the Hamiltonian will again be H (cyclical variation). The following postulate is imposed: it is impossible to extract work from the system in the state D by switching any external field in such a way. This is the statement of the second law in the Thomson formulation [45]. We shall see that this condition alone plus some companion ones are enough to derive the Gibbs distribution for the density matrix D of this system. Note that the condition presented does not impose irreversibility. Indeed, the evolution of the system remains purely unitary (thus reversible). As a result of the time-dependent field an external source has done the work [39]

$$W = \int_0^t \mathrm{d}\theta\, \mathrm{tr}\left[D(\theta)\frac{\mathrm{d}\mathcal{H}(\theta)}{\mathrm{d}\theta}\right] \tag{4.1}$$

$$= \mathrm{tr}[H(D(t) - D(0))] \tag{4.2}$$

where, when going from (4.1) to (4.2), we used integration by parts, and the equation of motion

$$\mathrm{i}\hbar\dot{D} = [\mathcal{H}(t), \rho(t)]. \tag{4.3}$$

Let us now introduce a unitary operator $V(t)$,

$$D(t) = \mathrm{e}^{-\mathrm{i}tH/\hbar} V D(0) V^\dagger \mathrm{e}^{\mathrm{i}tH/\hbar} \tag{4.4}$$

and rewrite equation (4.1) as

$$W = \mathrm{tr}[D(0)V^\dagger H V] - \mathrm{tr}[D(0)H]. \tag{4.5}$$

The quantity W is required to be positive. Since $W = 0$ for $V = 1$, we have to demand that $\mathrm{tr}[D(0)V^\dagger H V]$ is minimal for $V(1) = 1$. For V close to 1, one introduces an expansion

$$V = 1 + M + \mathcal{O}(M^2) \tag{4.6}$$

where $M^\dagger = -M$ and M is small. One obtains

$$W = \mathrm{tr}([D(0), H]M) + \mathcal{O}(M^2). \tag{4.7}$$

Since the sign of $\mathrm{tr}([D(0), H]M)$ can be arbitrary, $D(0)$ and H should commute for W to be minimal at $V = 1$. To obtain a more precise relation between the eigenvalues, we use a particular form

$$V = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{4.8}$$

when acting on the two-dimensional subspace formed by common eigenvectors $|i\rangle$ and $|k\rangle$ of H and $D(0)$, and $V = 1$ in the orthogonal complement of this subspace. Now it is easy to obtain

$$W = -(h_i - h_k)(r_i - r_k)\sin^2\theta \tag{4.9}$$

where h_i and h_k (or r_i and r_k) are the corresponding eigenvalues of H (or D). It is seen for any $|i\rangle$, $|k\rangle$ that

$$\text{if} \quad h_i \geqslant h_k \qquad \text{then} \quad r_i \leqslant r_k. \tag{4.10}$$

As follows from this, there is some positive non-increasing function f, such that

$$D = f(H). \tag{4.11}$$

The exponential form of f can be established from the standard reasons of extensivity. It is assumed that the form of the function f is (at least to some extent) universal, and does not depend on the Hamiltonian itself. Then for two interacting subsystems we have

$$\lim_{g\to 0} f(H_1 + H_2 + gH_{\mathrm{int}}) = f(H_1 + H_2) = f(H_1)f(H_2) \tag{4.12}$$

where gH_{int} is the Hamiltonian of the interaction. Under reasonable conditions f can be proved to be exponential:

$$\rho(H) = \frac{1}{Z}\exp(-\beta H), \tag{4.13}$$

with $\beta = 1/T \geqslant 0$.

5. Conclusion

The main purpose of this paper is to suggest a new viewpoint on possible connections between the thermodynamical and cosmological arrows of time. One aspect of such connections is well known and is based on the important role of special initial conditions for both arrows [2, 3]. However, the special initial conditions are by no means sufficient for generating the thermodynamical arrow of time: they are able to generate a pre-arrow of time only. Thus initial conditions cannot be the only part of the story. We show that the negative curvature of the Friedmann–Robertson–Walker universe and the effect of geodesic mixing can provide the condition necessary for the emergence of the thermodynamic arrow of time. Moreover, this mechanism can explain why CMB contains the major fraction of the entropy of the universe.

If this is indeed an origin of the thermodynamic arrow, then the thermodynamics in flat and positively curved universes need not be strongly time asymmetric, and the latter is observed since we happen to live in a universe with negative curvature. In particular, as we show in section 4, this does not mean that the second law of thermodynamics and the Gibbs distribution will have less chance to survive for non-negatively curved spaces, since these concepts do not necessarily depend on the thermodynamical arrow of time.

In other words, the symmetry of the Newtonian mechanics, electrodynamics, quantum mechanics and equilibrium statistical thermodynamics might purely survive in some universes. In this context the essence of the thermodynamical arrow must be understood as not the mere increase of entropy of an almost closed system, but as the fact that this arrow has a universal direction in the entire universe (see [46]). In the light of our suggested explanation of the emergence of this arrow, it may follow that the negative curvature is the very mechanism unifying all local thermodynamical arrows. According to this logic, in the flat or positively curved universes, i.e. in the absence of a global unification mechanism, there can be local thermodynamical arrows with various directions.

Another intriguing problem arising here is whether life can occur in such globally time-symmetric universes, or is the time asymmetry/negative curvature a necessary ingredient for the development of life—the *curvature anthropic principle*. The CMB has to carry the signature of this principle.

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References

[1] Gold T 1962 *Am. J. Phys.* **30** 403
[2] Penrose R 1979 *General Relativity: An Einstein Centenary Survey* ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
[3] Penrose R 1989 *The Emperor's New Mind* (Oxford: Oxford University Press)
Penrose R 1994 *J. Stat. Phys.* **77** 217
[4] Hawking S W 1985 *Phys. Rev.* D **32** 2989
[5] Page D 1985 *Phys. Rev.* D **32** 2496
[6] Petrosky T and Prigogine I 1997 *Adv. Chem. Phys.* **XCIX** 1
[7] Halliwell J J, Perez-Mercader J and Zurek W H (ed) 1996 *Physical Origins of Time Asymmetry* (Cambridge: Cambridge University Press) p 175
[8] Zeh H D 1992 *The Physical Basis of the Direction of Time* (Berlin: Springer)
Kiefer C and Zeh H D 1995 *Phys. Rev.* D **51** 4145

[9] Allahverdyan A E and Gurzadyan V G 2000 *The Chaotic Universe* ed V G Gurzadyan and R Ruffini (Singapore: World Scientific) p 228 (*Preprint* astro-ph/9910339)
[10] Gurzadyan V G 2001 Talk at *XXII Solvay Conference in Physics* (to appear in proceedings)
[11] Landau L D and Lifshitz E M 1978 *Statistical Physics* (Oxford: Pergamon)
[12] Gurzadyan V G and Kocharyan A A 1992 *Astron. Astrophys.* **260** 14
Gurzadyan V G and Kocharyan A A 1993 *Int. J. Mod. Phys.* D **2** 97
Gurzadyan V G and Kocharyan A A 1993 *Europhys. Lett.* **22** 231
Gurzadyan V G and Kocharyan A A 1998 *Quantum Gravity VI* ed V A Berezin, V A Rubakov and D V Semikoz (Singapore: World Scientific) p 542
[13] Gurzadyan V G and Kocharyan A A 1994 *Paradigms of the Large-Scale Universe* (New York: Gordon and Breach)
[14] Gurzadyan V G and Torres S 1997 *Astron. Astrophys.* **321** 19
[15] Gurzadyan V G 1999 *Europhys. Lett.* **46** 114
[16] Allahverdyan A E, Gurzadyan V G and Soghoyan A 1999 *Int. J. Mod. Phys.* D **8** 383
[17] De Bernardis P *et al* 2000 *Nature* **404** 955
De Bernardis P *et al* 2002 *Astrophys. J.* **564** 559
Netterfield C B *et al* 2001 *Preprint* astro-ph/0104460
Lee A T *et al* 2001 *Astrophys. J.* **561** L1
[18] Sievers J L *et al* 2002 *Preprint* astro-ph/0205387
[19] Baker G 2000 *Preprint* astro-ph/0006398, LAUR-00-2854
[20] White M, Scott D and Pierpaoli E 2000 *Astrophys. J.* **545** 1
[21] Trotta R, Riazuelo A and Durrer R 2001 *Phys. Rev. Lett.* **87** 231301-1
[22] Gurzadyan V G *et al* 2002 reported at *X ICRA Workshop Black Holes, Gravitational Waves and Cosmology (Pescara)*
[23] Perlmutter S *et al* 1999 *Astrophys. J.* **517**
[24] Riess A G *et al* 1998 *Astron. J.* **116** 1009
[25] Davies P C W 1976 *The Physics of Time Asymmetry* (Berkeley, CA: University of California Press)
[26] Allahverdyan A E, Balian R and Nieuwenhuizen Th M 2001 *Phys. Rev.* A **64** 032108
[27] Zwanzig R 1960 *J. Chem. Phys.* **33** 1338
[28] Jaynes E T 1957 *Phys. Rev.* **108** 171
Jaynes E T 1983 *Papers on Probability, Statistics, and Statistical Physics* ed R D Rosenkrantz (Dordrecht: Kluwer)
[29] Anosov D V 1967 *Comm. Steklov Math. Inst.* **90** 1
[30] Pollicott M 1992 *J. Stat. Phys.* **67** 667
[31] Collet P, Epstein H and Gallavotti G 1984 *Commun. Math. Phys.* **154** 569
[32] Lockhart C M, Misra B and Prigogine I 1982 *Phys. Rev.* D **25** 921
[33] Arnold V I and Avez A 1968 *Ergodic Problems of Classical Mechanics* (New York: Benjamin)
[34] Kubo R 1963 *J. Math. Phys.* **4** 174
[35] Mackay M C 1989 *Rev. Mod. Phys.* **61** 981
[36] Zaslavski G M and Sagdeev R Z 1988 *Introduction to Nonlinear Physics* (Moscow: Nauka) (in Russian: *Vvedenie v nelinejnuiu fiziku*)
[37] Allahverdyan A E and Saakian D B 1998 *Phys. Rev.* E **58** 1148
[38] Dynkin E B 1965 *Markov Processes* (New York: Springer)
[39] Balian R 1992 *From Microphysics to Macrophysics* vol I and II (New York: Springer)
[40] Lindblad G 1975 *Commun. Math. Phys.* **40** 147
[41] Schlögl F 1980 *Phys. Rep.* **62** 268
[42] Jarzynski C 1995 *Phys. Rev. Lett.* **74** 2937
[43] Lenard A 1978 *J. Stat. Phys.* **19** 575
[44] Pusz W and Woronowicz L 1978 *Commun. Math. Phys.* **58** 273
[45] Allahverdyan A E and Nieuwenhuizen Th M 2002 *Physica* A **305** 542 (*Preprint* cond-mat/0110422)
[46] Gell-Mann M and Hartle J B 1996 *Physical Origins of Time Asymmetry* ed J J Halliwell, J Perez-Mercader and W H Zurek (Cambridge: Cambridge University Press) p 311